



Manipulation of Graphs, Algebras and Pictures

Essays Dedicated to Hans-Jörg Kreowski
on the Occasion of His 60th Birthday

On Judgements and Propositions

Bernd Mahr

20 pages

On Judgements and Propositions

Bernd Mahr

Technische Universität Berlin, Germany

mahr@cs.tu-berlin.de

Abstract: This article studies some of the relevant and historically influential conceptions of the notions of 'judgement' and 'proposition' and discusses the relationship of these notions in these conceptions. In some detail the conceptions of Aristoteles, Brentano and Twardowski, Frege, Martin-Löf and of epsilon-theory are presented. A comparison of these conceptions shows fundamental differences which, in a way, illuminate the differences found in the architectures and formal appearances of logics.

Keywords: Judgement, proposition, assertion, presentation, interpretation, justification, truth, logic.

1 Introduction

Provoked by the different architectures and formal appearances of logics the question can be asked “*what the fundament of logic is that admits the different views on it.*” A natural though not obvious answer is that the different approaches to logic are all rooted on conceptions of judgement and proposition. Both notions have been a matter of dispute since the beginning of Greek philosophy and are still today under debate, and both are closely linked with logic. To study some of these conceptions and to thereby shed some light on the different forms of logic and on their relationship, is the author’s first motivation to write this article. It turns out, however, that this is not easy at all since the notions of judgement and proposition are deeply involved. They touch on and relate fundamental questions of language, ontology, psychology, philosophy and mathematics, and their meaning is far from being common sense. By some authors the notion of proposition is even objected to be meaningful at all, [vOQ80, pp. 331–401] and the word judgement is taken to express things of the most different kind, from the most elementary relation between the human mind and the world, [Kan90] up to what is realised by natural deduction proofs in intuitionistic type theory¹.

The author’s interest in the notion of judgement also has another reason. In his studies on the general notion of ‘model’² it was found that the key to resolve the problems, encountered in explaining the notion of ‘model’ in its full generality, is to transform the original question of “*what is a model*” from ontology to logic and to ask instead “*what justifies a judgement, that something is a model.*” Deeper analysis of this new question naturally leads to the notions of *conception* (in German *Auffassung*) and context. [Mah97] The notion of conception is closely related to the notions of *presentation* and *judgement*, which both have been extensively studied

¹See for example Göran Sundholm: *Proofs as Acts and Proofs as Objects: Some questions for Dag Prawitz*, as well as Prawitz’ response to these questions, both in [Han98, pp. 187–216, 283–337].

²See for example [Mah09].

in the 19th century and can count as sources of modern mathematics. On the other hand, *judgements* are based on *conceptions*. The author's studies of the notions of *conception* and *context* resulted in a 'model of conception,' [Mah10] which is an axiomatisation of reflexive universes of things relativised by their subject and context dependency. In a recent thesis [Wie09] set-theoretic realisations of this axiomatisation have been given and its consistency has been proven. Judgements and in particular the judgements of model-being have thereby a sound fundament in their prerequisite '*conception*.'

2 The Notions of Judgement and Proposition in Brief

By "*judgement*" one denotes both, the *act of judging* and the *result* of this act.³ A judgement, as an act and as the result of this act, is always concerning something, *that which is judged, the judged*. A conventional though rather imprecise definition states that *that which is judged* is a proposition, and that a *proposition* is what is *true* or *false*.

The traditional views on judgement date from the Presocratic philosopher Parmenides, [Par69] and, among others, from Plato in his *SOPHISTES*, and predominantly from Aristoteles. Aristoteles developed in his writings⁴, which were later collectively called *organon* (in German *Werkzeug*), in the sense of a *tool of the mind*, the first elaborated and most influential concept of judgement, and laid therewith the ground for what *logic* is about. It is common to the traditional views on judgement that that, which is judged, is affirmed or denied to *exist*, and that 'being,' 'the presence of being there,' [Tug03] and later also 'existence,' make the grounds for affirmation and denial. Judgements under these views concern things and matters in reality, and, accordingly, the notion of judgement under these views is based on reality as a fundament of truth. Despite many extensions and modifications much of the essence of these traditional views, namely of Aristoteles' approach to logic, has been maintained through the times until to the mid 19th century. And it can still be found today in the Tarski-style of semantics, as it is commonly used in the model-theoretic semantics of logic.

Major modifications on the traditional views, which finally lead to the formal treatment of logic today, originate from the work of George Boole, namely his *INVESTIGATION OF THE LAWS OF THOUGHT ON WHICH ARE FOUNDED THE MATHEMATICAL THEORIES OF LOGIC AND PROBABILITIES* [Boo58] from 1854, from the works of Bernhard Bolzano⁵, namely in his *WISSENSCHAFTSLEHRE* from 1837 and Franz Brentano [Bre08] in his *PSYCHOLOGIE VOM EMPIRISCHEN STANDPUNKTE* from 1874, as well as from the works of Brentano's students Kasimir Twardowski [Twa82], Alexius Meinong [Mei02] and Edmund Husserl [Hus93], namely on their conceptions of judgement. Maybe the strongest influence on modern logic had Gottlob Frege who, with his *BEGRIFFSSCHRIFT* [Fre07] from 1879, laid the ground for a new understanding of quantification and predication, and developed basic principles of the notion of judgement, which have later been widely adopted in the formal treatment of semantics. Influenced

³What actually is denoted by "judgement" depends heavily on what is considered to be an act of judging and what as the result of this act. See for example Göran Sundholm: *Proofs as Acts and Proofs as Objects: Some questions for Dag Prawitz*, as well as Prawitz' response to these questions, both in [Han98].

⁴These writings include *CATEGORIAE, DE INTERPRETATIONE, ANALYTICA PRIORA, ANALYTICA POSTERIORI, TOPICA* and *DE SOPHISTIIS ELENCHIIS*.

⁵[Bol81] from 1837, see also [Ber92].

from Frege’s work and other sources Alfred North Whitehead and Bertrand Russell wrote their *PRINCIPIA MATHEMATICA* [WR62] in the years 1910 to 1913 and Ludwig Wittgenstein responded in his *TRACTATUS LOGICO PHILOSOPHICUS* [Wit73], published in 1921, to Frege and to Russell’s *THEORY OF KNOWLEDGE* [Rus92], which Russell wrote in 1913 but then, in reaction to Wittgenstein’s criticism, abandoned. Later, in his *Philosophical Investigations* [Wit67], written in the time from 1945 to 1949, Wittgenstein recalled and criticised many of the thoughts he had put forward in the tractatus on the nature of language. The modern debates on the notion of judgement owe much to the theory of speech acts as it has been developed in the 50th and 60th of the last century first by John L. Austin [Aus02] and later by John R. Searle. [Sea08]

In the course of this development the notions of judgement and proposition became a subject matter in ontology and philosophy rather than in classical formal logic where they are, so to say, banned to the meta-level of formalisation and only implicitly present in the interpretation of sentences. Explicit use of the notion of judgement, however, can be found in Per Martin-Löf’s *INTUITIONISTIC TYPE THEORY*, published 1984, and in specification frameworks and formalisms inspired by him, like the calculus of constructions, LF, Coq or Isabelle, developed since 1992. Martin-Löf insists on a formal distinction between propositions and judgements. [ML84] This distinction he elaborates further in the lectures *ON THE MEANING OF THE LOGICAL CONSTANTS AND THE JUSTIFICATIONS OF THE LOGICAL LAWS*, which he published in 1996, and in a recent lecture on *ASSERTIONS, ASSERTORIC CONTENTS AND PROPOSITIONS*⁶ in 2008. Martin-Löf’s careful considerations also influenced some of the conceptualisations in epsilon-theory (see section 9 below), as it is being studied by the author and his co-workers.

3 Realistic Conceptions of Proposition and Judgement

At the very beginning of his *DE INTERPRETATIONE* Aristoteles explains his view on the relationship between language, mind and reality, which is essential for the understanding of his conception of judgements: “*that what is expressed* [logos, in German *Satz*, in English *sentence*] *is a symbol of the states of the soul, and that which is written is a symbol of that which is expressed*” [...] “*that, of which the states of the soul are images, are the things.*” [TW04, p. 19] What is to observe here is that the states of the soul, which may be understood as *thoughts* in the sense of mental states, mediate between reality on the one side, and sentences being expressed and written on the other.

In his conception of the notion of judgement Aristoteles takes first of all a linguistic view, but combines it with a psychological and an ontological perspective. He describes a judgement as to being a particular type of sentence: “*Though a sentence is meant to denote, not every sentence is a judgement but only one in which the assertion of truth or falseness is present. It is, however, not present in every sentence since, for example, a wish is a sentence, but it is neither true nor false.*” [Ari67b, p. 7] Today we would say that he distinguishes different kinds of sentences, a distinction which in speech act theory is made by the distinction of illocutionary forces in the acts of ‘expressing’ by means of sentences. As judgements he singles out assertoric

⁶See [ML96, pp. 11–60]. His recent thoughts on judgements Martin-Löf presented in the lecture *Assertions, Assertoric Contents and Propositions*, which he gave at the workshop on *Judgements, Assertions, and Propositions - The Logical Semantics and Pragmatics of Sentences* at TU Berlin on January 11, 2008.

sentences and as the criterion for a sentence to be a judgement he states that the sentence is to be grammatically composed of two parts: a subject and a verb. Both parts, in his view, have meaning in the sense that they both denote something by convention. And as the criterion for the truth of a judgement he states that in the composition of subject and verb, the copula “is” or a derived form of it, has to reflect the relation of the things the subject and the verb denote: “*To affirm [katáphasis] is to express something towards something, and to deny [apóphasis] is to express something away from something*”; and concerning truth and falseness of a judgement made, Aristoteles writes: “*The one, who thinks as being separated what is separated and as being composed what is composed, thinks true; but he thinks false, whose thoughts are contrary to the things.*” [Ari67c, p. 7] It is to note here that in Aristoteles’ view ‘true’ and ‘false’ are not atomic values, as which they are taken by Frege, but qualities of thinking. Other than logicians today Aristoteles restricts his observations to simple judgements of the subject-predicate form. He does not consider complex judgements, which are composed of sub-judgements and logical connectives. For the meaning of simple judgements he follows a simple variant of the *principle of compositionality*, which postulates that the meaning of composed expressions is the composition of the meanings of the individual expressions. This principle is attributed to Frege who applied it in his functional interpretation of sentences.⁷ A similar principle has also been stated by Leibniz in his *ARS CHARACTERISTICA* [Krä88, p. 105].

In Aristoteles’ view there are two qualities of judgements, namely affirmation and denial, and “*every judgement is either a judgement about what there is in reality, what there is by necessity or what there is by possibility*”. Aristoteles also distinguishes three kinds of judgement: “*a judgement [...] is either general, or particular or undetermined. General means that something applies⁸ to all or none, particular means that something applies to a single, or to a single not, or not to all, and undetermined means that it applies or does not apply without determination of the general or the particular ...*” [Ari67a] With the concepts of quantification in modern predicate logic, where sentences have a complex structure, the notions of *general* and *particular* have become more precisely formulated.

4 Formalisation of Categorical Judgements

Today we would rephrase Aristoteles’ view on judgements as follows: sentences and their written forms refer to thoughts created in a mental act, and these thoughts as mental states, are images of existing things. Truth is assigned to thoughts, in the sense of a mental act, and requires the correspondence between that which is thought and the things of which the thoughts, in the sense of mental states, are images. Though thoughts, in Aristoteles’ sense, correspond to what in speech act theory is called a propositional act, there is also a major difference: Aristoteles does not think of thoughts in terms of reference and predication in the way we do, but he insists in, to say it in modern terms, a type theoretical reading of thoughts. Take as an example the sentence “all men are mortal,” which, according to his grammatical criterion for judgements and his classification of kinds, is a general judgement. Aristoteles’ reading of this sentence can be

⁷See for example [Fre66, pp. 72–91].

⁸The English word ‘applying’ is used here to translate the Greek word ‘hyparchein.’

formalised as the type proposition

(all men) are (mortal)

in which the composition of two things is asserted. This sentence is true if in the real world *mortality* (which is the thing whose image is symbolised by the written expression ‘mortal’) applies to *all humans* (which is the thing whose image is symbolised by the written expression ‘all men’). The famous ‘problem of universals’ in the middle ages was about the question if things as expressed by the words ‘all men’ and ‘mortal,’ have existence. Since Frege, motivated by the concept of a mathematical function in arithmetic and higher analysis, proposed predication as a form of function application, which results truth-values, [Fre75, pp. 17–39] and since he introduced individual variables for the indication of individuals in quantification, [Fre75, p. 33] today’s conventional reading of the respective sentence⁹ can be expressed in the formalisation

$$\forall x.(\text{men}(x) \rightarrow \text{mortal}(x)).$$

If we write atomic predication in the form of a type proposition [Mah93], we get

$$\forall x.(x : \text{men} \rightarrow x : \text{mortal})$$

This type propositional formalisation shows, on the one hand, a closer similarity to the Aristotelian view, though the sentence as a whole is not a simple judgement anymore. It has a complex structure. It also shows a close similarity to the truth condition for atomic predications in the Tarski-style of model-theoretic semantics, which, in a semiformal style, is phrased as

For all h it is true that, if $(h \in A_{\text{men}})$ then $(h \in A_{\text{mortal}})$

where h denotes an unspecified individual in the domain of interpretation, and A_{men} and A_{mortal} denote subsets of this domain. It is interesting to note that the model-theoretic truth condition expresses the same idea as the truth condition in Aristoteles’ view: “the presence of being there.” The only difference is that the Aristotelian truth condition concerns things in reality while in the Tarski-style of model-theoretic semantics the condition is expressed, relative to a given domain of interpretation, in terms of set-theoretic membership. It is also interesting to note that the (semantic) reading of $\forall x.$ as “for all h , which instantiate x , it is true that” turns the sentence

$$\forall x.(x : \text{men} \rightarrow x : \text{mortal})$$

into a symbolisation of a proposition in which an h -indexed family, not of propositions, but of judgements is expressed.

Following the conventional model-theoretic interpretation, the above sentence is also true in a world where humans cannot be found, since in this case for every instantiation of x the premise of the implication is false. This property indicates that the use of variables in quantification resolves the difficulty Parmenides saw in the notion of ‘non-existence’: He concluded that negation cannot be thought of because “what is not is not, and can therefore not be.” [Par69] In the modern understanding of ‘non-existence,’ instead, non-existence is the property of the domain

⁹See also Russell’s *On Denoting* from 1905, which in German translation appeared as [Rus00, pp. 3–22].

of interpretation that the thing with the property in question cannot be found in it, i.e. is not there. The ‘logical’ reading [TW04] of a simple judgement as a complex sentence resolves this difficulty because it reads “existence” as “existence with some property.” But this conception of existence as ‘being there’ has the consequence that, before existence can be asserted, there is always a universe needed to be given, which consists of the things that ‘are there’ and are known to fall into defined categories. In conventional logic this is enforced in the inductive definition of formulas and the set-theoretic structures for the interpretation of formulas. If we read, to use an example of Quine, the “*existence of unicorns*” as the “*existence of something which unicorns,*” the question comes up of what the nature of this something is. At the linguistic level of conventional logic it is a variable and at the semantic level in the Tarski-style interpretation it is possibly any element in the domain of interpretation. But what it is in a reality that can hardly be well-defined as a set, and what it is itself if it cannot be given the ontological status of a real thing but only the status of an intentional object, is even today a rather open question.¹⁰

5 Brentano’s Notion of Judgement

Early conceptions of judgement and proposition with a particular emphasis on their roles in science and logic have been studied by Bernhard Bolzano in his *WISSENSCHAFTSLEHRE* (1837). Bolzano laid the ground for further investigations by Brentano and by his students. A thorough account of these investigations can be found in the article *AUSTRIAN THEORIES OF JUDGEMENT: BOLZANO, BRENTANO, MEINONG, AND HUSSERL* by Robin D. Rollinger. [Rol08, pp. 233–261] Of particular interest here is Brentano’s conception of intentionality. It opened a new perspective on judgements. Following Searle, intentionality is also constituent to speech acts. [Sea04, pp. 5–13] Searle therefore develops his theory of intentionality as a foundation for his theory of speech acts. Speech act theory¹¹ strongly influenced modern conceptions of the linking of mental acts and symbolic presentations, and is therefore fundamental for intuitionistic approaches to the notions of *proposition* and *judgement*.

In Brentano’s *PSYCHOLOGY FROM AN EMPIRICAL STANDPOINT* (1874), an act of judging is a case of a mental act. According to his classification, all mental acts are either presentations [in German *Vorstellung*], or judgements or volitions, and all contain an object intentionally within themselves. This can be concluded from the following famous citation: “*Every mental phenomenon is characterised by what the Scholastics of the Middle Ages called the intentional (or mental) inexistence of an object, and what we might call, though not wholly unambiguously, reference to a content, direction towards an object (which might not to be understood here as meaning a thing), or immanent objectivity. Every mental phenomenon includes something as object within itself, although they do not all do so in the same way. In presentation something is presented, in judgement something is affirmed or denied, in love loved, in hate hated, in desire desired and so on. This intentional in-existence is characteristic exclusively of mental phenomena. No physical phenomenon exhibits anything like it. We would, therefore, define mental phenomena by saying that they are phenomena, which contain an object intentionally within*

¹⁰I consider this a serious question. A somewhat unsatisfactory answer is given in [Gro92, pp. 106–119]. Grossmann considers intentional relations as abnormal relations because their objects need not to have existence.

¹¹For the history of speech act theory see [Smi90, pp. 29–61].

themselves.” [Bre95, pp. 88–89] What in a judgement is affirmed or denied is, in Brentano’s view, the existence of this object. And so we might say that Brentano conceives of a proposition as the ‘*existence of the object of a judgement*,’ which may be the case or not.

Brentano’s conception of *intentionality* has strongly influenced modern philosophy, namely logic, ontology, existential philosophy and theories of language and semantics. He not only claims that every mental act is a presentation or rests on a presentation, but also that a distinction has to be made between the object presented and the content of its presentation. Bernhard Bolzano was convinced that there are presentations, which, though they have content, have no object, [Bol81, §67, pp. 304–306] like the presentation of a golden mountain or the presentation of a round square. This conviction, however, has the consequence that such things cannot be judged to not exist. But this appears to be against our intuition as we can think or even have an imagination of things, which, we know, do not exist, and also can judge that they do not exist. We can think of a round square and even imagine unicorns and a flat earth, and we daily have the presentation of a user-friendly computer system. Even further, we seem to need a presentation of something before we can assert it to not exist. This observation, it turns out, touches at a major problem, the question of what exactly we mean by a proposition and by a judgement and how we understand the relation between a judgement and ‘its’ proposition. The disputes in analytic philosophy literature show that even today this problem has not yet found a commonly accepted solution.

6 Twardowski’s Theory of Presentations

Kasimir Twardowski, one of Brentano’s students in Vienna, addressed this problem in his Habilitation thesis ON THE CONTENT AND OBJECT OF PRESENTATIONS - A PSYCHOLOGICAL INVESTIGATION [Twa77] in 1894. He studied the concept of *presentation* (in German *Vorstellung*) and focuses on the observation that presentations imply in what they present to the mind, two different objects rather than one: the object towards which a presentation is directed (in German *Gegenstand*), and the object, which is its content (in German *Inhalt*). Though the topic of his thesis is the concept of presentation, he also deals with the notion of *judgement* (in German *Urteil*) and sees a “*perfect analogy*” [Twa77, p. 7] between *presentations* and *judgements*. Both, he states, imply an act, both concern something, namely what is presented and what is judged, and in both this something, which is presented or judged, is to be subdivided into object and content, the latter of which he calls the intentional object of the act.

While one and the same object can be presented as well as being judged, he finds the distinction between presentation and judgement in the intentional object of the act: “*When the object is presented and when it is judged, in both cases there occurs a third thing, besides the mental act and its object, which is, as it were, a sign of the object: its mental ‘picture’ when it is presented and its existence when it is judged.*”¹² Here the ‘object’ of a judgement is the object about which the judgement is made, while the ‘subject’ of a judgement is that what is affirmed or denied, *the object’s existence*. Twardowski insists that “*presentation and judgement are two separated classes of mental phenomena without intermediate forms of transition.*” [Twa77, p. 6]

¹²See [Twa77, p. 7]; the conception of ‘existence’ as the content of a judgement is not obvious. See Grossmann’s criticism on this conception in: Reinhard Grossmann: *Introduction*, in [Twa77, pp. VII - IVXXX, here pp. IX - XI].

The distinction between object and content in what is presented or judged is most natural, though the question of what exactly an object is, about which a judgement can be, still remains unanswered. In the beginning of his treatise on objects in §7 of his investigation Twardowski gives a partial answer to it: “*According to our view, the object of presentations, of judgements, of feelings, as well as of volitions [in German ‘Wollungen’], is something different from the thing as such [in German ‘Ding an sich’], if we understand by the latter the unknown cause of what affects our senses. The meaning of the word ‘object’ coincides in this respect with the meaning of the word ‘phenomenon’ or ‘appearance,’ whose cause is either, according to Berkeley, God, or, according to the extreme idealists, our own mind, or, according to the moderate ‘real-idealists’ the respective things as such. What we have said so far about objects of presentation and what will come to light about them in the following investigations is claimed to hold no matter, which one of the just mentioned viewpoints one may choose. Every presentation presents something, no matter whether it exists or not, no matter whether it appears as independent of us in our own imagination; whatever it may be, it is - insofar as we have a presentation of it - the object of these acts, in contrast to us and our activity of conceiving [in German ‘vorstellenden Tätigkeit’].*” [Twa77, p. 33]

And at the end of his treatise on objects he writes: “*Summarizing what was said, we can describe the object in the following way. Everything that is presented through a presentation, that is affirmed or denied through a judgement, that is desired or detested through an emotion [in German ‘Gemüththätigkeit’], we can call an object. Objects are either real or not real; they are either possible or impossible objects; they exist or do not exist. What is common to them all is that they are or that they can be the object (not the intentional object!) of mental acts [in German ‘psychischer Akte’], that their linguistic designation is the name ..., and that considered as genus [in German ‘Gattung’], they form a summum genus, which finds its usual linguistic expression in the word ‘something’ [in German ‘etwas’]. Everything which is in the widest sense “something” is called “object,” first of all in regard to a subject, but then also regardless of this relationship.*”

An important term in this citation is the word “through.” It assigns the mental act and its intentional object the role of a mediator: *through* the act and content of a presentation an object is presented, and, accordingly, *through* the acts of affirmation or denial of existence an object is judged. Every object, now, existing or not, can be seen to be the object of both, a presentation and a judgement. Twardowski’s concept of object of a mental act solves the above mentioned problem of judging non-existing objects to not exist, and Bolzano’s belief in presentations, which have no object, turns out to be wrong: “*The confusion of the proponents of objectless presentations consists in that they mistook the non-existence of an object for its not being presented.*” [Twa77, pp. 20–29, here p.22] But despite the fact that the general approach to the objects of a judgement seems to be most reasonable, the ontological status of objects in intentional relations is subject to controversies and not at all free from problems. It is therefore heavily debated in the literature. Progress has been made with the invention of ‘states of affairs’ and with the conception that judgements concern states of affairs rather than objects.¹³ It seems that the invention of states of affairs has two sources: *predications* on the one hand, as they have been used by Frege in his BEGRIFFSSCHRIFT for the purpose of formalising arithmetic and by Peano and Russell who applied and developed formal description techniques for other parts of

¹³See Grossmann’s introduction to Twardowski in [Twa77] and [Gro92].

mathematics, and “*Sachverhalte*” on the other, as certain types of (intentional) objects, studied by Meinong, Husserl and Reinach [Smi89]. This view is later also found in Wittgenstein’s *Tractatus*, the first two sentences of which are: “*Die Welt ist alles, was der Fall ist. Die Welt ist die Gesamtheit der Tatsachen, nicht der Dinge*” (“The world is everything that is the case. The world is the total of what is the case, not of the things”).¹⁴

7 Frege’s Conception of Proposition and Judgement

An answer of pragmatic value for the questions of what propositions and judgements are and how they are related seems possible only within a prescriptive deductive or semantic framework¹⁵. Conventional formal logic makes no clear distinction between the two concepts and avoids their conceptualisation at all. And a dedicated formal theory of propositions and judgements has not yet been proposed. However, there are considerations, which aim at clarification.

In his article *ÜBER SINN UND BEDEUTUNG* (1892) Gottlob Frege discusses the meaning of verbal expressions, like names, denotations and sentences, and draws the well known distinctions between *sign* (in German *Zeichen*), *sense* (in German *Sinn*), *reference* (in German *Bedeutung*), and *presentation* (in German *Vorstellung*). “*A sign is the expression of some sense and it denotes or references its reference.*” [Fre75, p. 46] A comparison of this distinction with Twardowski’s distinction of *names*, *content* and *object* of a presentation shows many similarities, but also major differences: Frege’s sense is not part of a mental state or act. It has objectivity. Therefore *presentations* are not *senses* and therefore Twardowski’s *content* is not the same as Frege’s *sense*, even though they play similar roles in the designation of an object. And *names* in Twardowski’s conception do not designate matters of affairs but objects. In Frege’s conception also sentences have a *sense* and a *reference*, and the *sense of a sentence* is what he calls a *thought* (in German *Gedanke*). Frege’s concept of thought is what Husserl and (the early) Wittgenstein call *matter of affairs* [TW04, p. 17], and what Russell calls *proposition* for which he later uses the word *assertion*.¹⁶ If a *sign* is a *sentence*, the question is what it references. In Frege’s view, a sentence references a *truth-value*, i.e. the value *true* or *false*. Accordingly, also truth-values are objects (in German *Gegenstände*). Since in Twardowski’s Habilitation there is no citation of Frege’s work, we must conclude that Frege’s work was not known in Vienna at that time. Frege’s conception of proposition was later adopted in formal logic, though in the hidden form of the recursive definition of interpretation and validity, which is derived from Frege’s principles of compositionality and truth functionality. In view of its pragmatic language use and meaning, however, Frege’s notion of sense has been strongly criticised.¹⁷

Frege also made an important contribution to the conceptualisation of *judgement*. What is being affirmed or denied in a judgement is that a proposition is true or false, or in other words, that a matter of affairs is a fact or not. Frege thereby frees the concept of judgement from its binding to object-existence. He also draws a clear distinction between proposition and judgement

¹⁴See [Wit73, p. 11], English translation by the author.

¹⁵To a certain degree this is done in Martin-Löf’s intuitionistic type theory and formalisms following him (see below). See also [HHP93, pp. 143–184].

¹⁶See also [ML96, pp. 11–60], where he gives an account on the development of the concepts of proposition and judgement in the light of his intuitionistic type theory.

¹⁷See for example [Dum82].

by saying that a judgement is not just the affirmation or denial of a proposition, but that the affirmation or denial is asserted. In his *BEGRIFFSSCHRIFT* (1879) Frege introduces a notation for assertions, the vertical stroke, which he later combined with the horizontal stroke to indicate assigning truth, and the negated horizontal stroke to indicate falseness. So, for example, the assertion that the earth is flat can then be expressed as

$$\vdash \text{flat (earth)}$$

and the assertion that the earth is not flat can be expressed as

$$\nvdash \text{flat (earth)}$$

Here the symbol \vdash is to be read as “it is asserted that it is not the case that,” and not as “it is not asserted that it is the case that” which would be the negation of the assertion. Frege’s observation that a judgement is more than just the statement of a true or false proposition, because a statement could also mean an assumption, makes the distinction between different kinds of judgements, as it was customary in the traditional views on judgements, meaningless. If we respect this observation, a judgement is always an affirmation. Using truth predicates, written in type propositional form, the above assertions may be read as affirmations of the sentences

flat (earth) : true

and

flat (earth) : false

Frege gives an impressive insight into his style of writing and the purpose and use of formal notations in mathematics in *ÜBER DIE WISSENSCHAFTLICHE BERECHTIGUNG EINER BEGRIFFSSCHRIFT* (1882). He motivates the notation of the judgement stroke \vdash with the pragmatic needs in the writing of formal expressions and in the depiction of logical derivations on a sheet of paper. The question of “*how can we write?*” becomes prominent and the analysis of “*what can we write down?*” leads to the new view on judgements. Frege uses the judgement stroke in a given context of discourse, the context of a given system of axioms and rules or of a given model or theory. It is this context, which justifies the assertion of truth.

8 Martin-Löf’s Conception of Judgement and Proposition

In his *INTUITIONISTIC TYPE THEORY*, Martin-Löf makes the following distinction between proposition and judgement: “*Here the distinction between proposition (Ger. Satz) and assertion or judgement (Ger. Urteil) is essential. What we combine by means of the logical operations (falsum, implication, and, or, for all, there is) and hold to be true are propositions. When we hold a proposition to be true, we make a judgement:*

((A : proposition) is true) : judgement

“*In particular, the premises and the conclusion of a logical inference are judgements. The distinction between proposition and judgement was clear from Frege to Principia. These notions*

have later been replaced by the formalistic notions of formula and theorem (in a formal system), respectively. Contrary to formulas, propositions are not defined inductively. So to speak, they form an open concept. In standard textbook presentations of first order logic, we can distinguish three quite separate steps:

1. Inductive definition of terms and formulas
2. Specification of axioms and rules of inference
3. Semantical interpretation

“Formulas and deductions are given meaning only through semantics, which is usually done following Tarski and assuming set theory.

“What we do here is meant to be closer to ordinary mathematical practice. We will avoid keeping form and meaning (content) apart. Instead we will at the same time display certain forms of judgement and inference that are used in mathematical proofs and explain them semantically. Thus we make explicit what is usually implicitly taken for granted. When one treats logic as any other branch of mathematics, as in the metamathematical tradition originated by Hilbert, such judgements and inferences are only partially and formally represented in the so-called object language, while they are implicitly used, as in any other branch of mathematics, in the so-called metalanguage.

“Our main aim is to build up a system of formal rules representing in the best possible way informal (mathematical) reasoning.” [ML84, pp. 3–4]

In Martin-Löf’s informal reasoning by means of formal rules judgements are not viewed from a language perspective, as Aristoteles did and as we still do today, at least in most of the philosophical and formal logic accounts, but are closer to speech acts in the sense of Austin’s “*how to do things with words.*” Martin-Löf’s informal reasoning is to be seen as a performing of acts of judging, which consist in the writing down of judgements. The writing down of judgements is justified by the rules of the type system, whose premises are again judgements. Some rules, however, have no premises. They are axioms. Judgements in Martin-Löf’s type theory have one of the following written forms: A set, $A = B$, $a \in A$, or $a = b \in A$. The last two of these forms correspond closely to the judgements to be made in Cantor’s criterion for a set to be ‘well-defined,’ which he phrased in 1882, with the study of powers, when he refined his notion of a set¹⁸: “*I call an aggregate (a collection, a set) of elements which belong to any domain of concepts [in German *Begriffssphäre*] well-defined, if it must be regarded as internally determined on the basis of its definition and in consequence of the logical principle of the excluded middle. It must also be internally determined whether any object belonging to the same domain of concepts belongs to the aggregate in question as an element or not, and whether two objects belonging to the set, despite formal differences, are equal to one another or not.*”

All forms of judgement in Martin-Löf’s type theory propose a natural set-theoretical interpretation. The given system of rules, however, admits also other readings of these forms. One of these readings corresponds to the well known concept of ‘propositions as types,’ also known as

¹⁸The criterion is phrased in a letter by Cantor to Richard Dedekind in 1882; see for example [Dau79], cited in English from [Dau79, p. 83].

the Curry-Howard isomorphism, and reads the judgement $a \in A$ as “*a is a proof for the proposition A.*” This reading is not only the basis of his system as an intuitionistic theory of types, but is also consistent with an intuitionistic interpretation of his approach as a whole: From a meta-level perspective the written forms of judgements symbolise propositions for which his system lays down what counts as a proof.¹⁹ This is the way how he explains semantically these forms of judgements.

Concerning propositions Martin-Löf writes: “*Classically, a proposition is nothing but a truth value, that is, an element of the set of truth values, whose two elements are the true and the false. Because of the difficulties of justifying the rules for forming propositions by means of quantification over infinite domains, when a proposition is understood as a truth value, this explanation is rejected by the intuitionists and replaced by saying that*

A proposition is defined by laying down what counts as a proof of the proposition,

and that

a proposition is true if it has a proof, that is, if a proof of it can be given.

“*Thus, intuitionistically, truth is identified with provability, though of course not (because of Gödel’s incompleteness theorem) with derivability within any particular formal system.*” [ML84, p. 11]

The conventional conception of formal logic leaves these notions of proposition and judgement out of its consideration. It treats these notions only implicitly in the recursive definitions of interpretation and avoids their explicit notation.

9 Logics with Propositional Variables

Also classical propositional and predicate logics can be seen as conceptions of propositions. They provide linguistic means, usually in terms of alphabets and inductive definitions, to write sentences, which through interpretation become either *true* or *false*. Sentences in propositional logic are built up from propositional variables and propositional connectives like ‘and,’ ‘or,’ ‘not,’ and may be others. The interpretation of propositional sentences is based on a given truth-assignment, which assigns truth-values ‘*true*’ or ‘*false*’ to propositional variables, and is defined by induction as an evaluation function, which assigns truth-values to propositional sentences. Here the principles of compositionality and truth functionality are maintained in their purest form. The ‘architecture’ of (first order) predicate logic is not much different, except that atomic formulas are not propositional variables but *predications* and *equalities*, that variables are object-variables taking values from a given semantic domain, and that expressive power and expressiveness of predicate logic are enriched by the use of function symbols for object description, relation symbols for predications and quantifiers ranging over the elements of the carrier sets of the semantic domain.²⁰ Sentences in these logics are complex forms, which express, through their

¹⁹The status of a proof in the intuitionistic conception of truth has been a matter of discussion. See for example [Sun94], as well as Sundholm’s and Prawitz’ debate in the above mentioned volume 64 in Theoria. [Han98]

²⁰See for example [EMC⁺01, pp. 221–455].

interpretation for a given truth-assignment or in a given semantic domain, *sense*, to use Frege's terminology. They may also be read as formal statements of matters of affairs, which are implicit in their formal interpretation. But these matters of affairs are usually not made explicit and only hidden in the recursive interpretation of sentences. The processes of interpretation then only yield truth-values, and equivalence at the object-level can only be expressed in terms of 'having the same truth-value,' rather than 'stating the same matter of affairs.' This is, how classical logic with a strict separation of syntax and semantics avoids the notions of proposition, and how it treats judgements only implicitly in its definition of the interpretation process relative to a given truth-assignment and semantic domain.

There is a particular logic to explicitly express truth of propositions, quantification over propositional variables and propositional equivalence, which has been developed by Werner Sträter in [Str92] and is called \in_T -logic. One of the motivations for its design was to avoid partial truth predicates and to admit formulations like the liar paradox

$$x \equiv x : \text{false}$$

to be treated as contradictions. \in_T -logic grew out of an extensional interpretation of types,²¹ which reads a *type proposition*

$$e : T$$

as a *statement of membership*

$$[[e]] \in [[T]]$$

The type proposition $\varphi : \text{true}$ would then be read as a statement of membership with $[[\varphi]]$ denoting a proposition and $[[\text{true}]]$ a set of true propositions.

\in_T -logic is equipped with propositional constants, variables and connectives, quantification over propositional variables, truth predicates and propositional equivalence. Its semantics is defined in the Tarski-style, where the semantic domain is a domain of propositions and the interpretation function ensures the natural properties of propositional and of first order logic, as far as they apply. It fulfils the well known Tarski biconditionals in the sense that the sentence

$$\forall x.(x : \text{true} \leftrightarrow x) : \text{true}$$

is universally true. \in_T -logic has an impredicative nature and allows for intensional semantics of its sentences. Extensions of this logic have been defined and studied by Philipp Zeitz [Zei00], who introduced Parameterization, by Sebastian Bab [Bab07], who extended \in_T -logic by modal operators, and by Steffen Lewitzka [Lew09], who studied an intuitionistic variant of \in_T -logic.

Also Frege defines in his BEGRIFFSSCHRIFT²² a logic with propositional variables. Frege's notations admit the reference to objects and to functions over objects, as well as to functions over functions. They allow for propositional variables ranging over truth-values, which are viewed as being objects like any other object, they admit to write operators, which cover the classical

²¹See [MSU90] and [Mah93].

²²See also [Her83, pp. IX–XV].

propositional connectives, and they include propositional equality and quantification. The expressiveness of Frege's logic is closely related to a certain instance of Parameterized \in_T -logic in the sense of Zeitz. However, there is no perfect analogy. The major difference is in the use of quantification, and in the style of semantics.

Intuitively, there is good reason to also view classical logics as theories of propositions, no matter if in these logics propositions form a distinct and well defined category of entities to be dealt with or not. This is obvious in the case of logics, which admit propositional variables, and it is even more obvious for \in_T -logic and its extensions, which explicitly support propositional quantification and equivalence, and assume propositions as elements of their semantic domains. Can the same be said for judgements? Classical logics introduce notations for judgements at their meta-level, usually in the form of a sign denoting validity of a sentence under a given interpretation, like for example the validity of φ under the truth-assignment B

$$B \models \varphi$$

But they do this in a rather propositional manner, as they also allow to denoting invalidity.

$$B \not\models \varphi$$

They treat judgements as propositions at the meta-level. Otherwise there is little difference between these signs and Frege's judgement stroke. Frege's notation is based on the assumption of a given model so that there is no need to indicate the truth assignment or the semantic domain of interpretation. And also the fact that the judgement stroke is written at the object-level of formalisation is not of much relevance, since it is used at this level not as an operator but as an indicator and, in addition, only at the outermost position of the two-dimensional expressions. The judgement stroke can be omitted but it cannot be negated. Negation of the judgement stroke would turn it into a propositional operator. This, by the way is the reason why the expression φ : true in \in_T -logic cannot reasonably be interpreted as a judgement. The judgement stroke is not subject to interpretation but is a sign, which has a purely pragmatic meaning. It indicates what is an answer to the question "What can I write?"

10 Summary

To turn to the question of how the notions of judgement and proposition discussed do relate to each other we try to answer the following questions:

1. Is there a distinction made between assertion and judgement?
2. Is there a distinction made between proposition and judgement?
3. What is that which is expressed by a judgement?
4. How is a judgement justified?

1. In his INTUITIONISTIC TYPE THEORY Martin-Löf speaks of judgements rather than assertions, and in his recent lecture on ASSERTIONS, ASSERTORIC CONTENTS AND PROPOSITIONS, [ML08] he speaks of assertions rather than judgements. Not fully conform to other

naming conventions he uses the term *assertion* to denote the verbal expression of a judgement, in the form of a spoken or written sentence, and with the use of some language or notational convention. But, as he argues, the interchangeable use of the terms judgement and assertion is justified since in logic both depend on rules, the focus of his interest, and rules are the same for both. Following the assumption that assertions are verbal expressions, judgements may be seen as the mental counterparts of assertions. But this view can hardly be maintained since also assertions include an act of judging. Aristoteles avoids this problem by distinguishing between mental states on the one side, which stand in an image-relation to the things, and judgements as symbolisations of these states on the other. He assumes, at least implicitly, that there are two acts: the act of thinking, which, as he says, can be true or false, in the sense of right or wrong, and the act of symbolising, which produces a sentence or its written symbolisation. Brentano and Twardowski discuss judgements solely at the level of mental acts. Written forms are out of their interest. In their understanding contents are in the mind while objects are embedded in an intentional relation. The ontological status of objects, however, remains somewhat unclear²³. They may be real or mental objects, like thoughts, and may exist or not. Frege's thoughts, instead, are explicitly thought of as being independent from some mind, as they have objectivity and can be shared by several subjects. In \in_T -Logic the distinction between judgement and assertion is mostly irrelevant, like in most formal logics with a set-theoretic Tarski-style of semantics. Sets in a set-theoretic universe of interpretation have the same ontological status as thoughts in the sense of Frege. [Gro92, pp. 106–119] They are the means by which, through the application of rules of interpretation, sense and reference, in the sense of Frege, are being determined as elements of the given universe.

2. Not in all the conceptions discussed a distinction between proposition and judgement is being made. In Aristoteles' conception there is an act of thinking, which has all the ingredients of an act of judging, while the forms of sentences, which are called judgements, may be understood as the grammatical forms of propositions, and the mental states to which they refer, may be understood as propositions, which can be true or false. In Brentano's and Twardowski's conception of judgement the concept of proposition is not explicitly named. What a judgement is about is an object and not necessarily something which is true or false, but something which exists or not. Only if we read the content of a judgement, which is the *existence of the object the judgement is about*, as a kind of proposition, we find a conception of proposition with the familiar relation between propositions and judgements. Separated from the object to which the judgement refers, is its presentation on which the judgement relies and whose content is a mental image. If the objects of judgements are matters of affairs, [Gro92] rather than objects of some other kind, Brentano's and Twardowski's conception of judgement includes much of the conception of judgement we use today. But still then there are major differences to Frege's conception. Frege's sense has objectivity, while contents of judgements, in Brentano's and Twardowski's sense, are mental images and are therefore of subjective nature. Frege's sense is a thought, which objectively represents a matter of affairs, but what a judgement refers to is not this matter of affair but a truth-value. The two approaches would be easier to compare, if we could distinguish between that which a judgement *is about* and that what the sentence expressing the judgement *refers to*. A judgement would then be about a matter of affairs (Frege's *sense* and Brentano's and

²³See the introduction to [Twa77].

Twardowski's *content of the presentation underlying the judgement*) and refer to a truth-value (Frege's *reference* and Brentano's and Twardowski's *intentional object*). This distinction would be rather natural because of matters of affairs it can meaningfully be said that they exist or not, depending on whether they are facts or not, and this distinction would also be closer to what Wittgenstein proposes in his TRACTATUS LOGICO PHILOSOPHICUS, [Wit73] but it would not be fully consistent with Frege's and Brentano's and Twardowski's original conceptions.

In his INTUITIONISTIC TYPE THEORY Martin-Löf draws a clear distinction between propositions and judgements and gives formal rules for the formation of judgements. In \in_T -Logic, however, the notion of judgement remains only implicit, like in other conventional logics. While the elements of the domain of interpretation are explicitly assumed to be propositions, whatever form they have, the notion of judgement, in the sense of Martin-Löf, is in \in_T -Logic only present at the meta-level and not part of the 'object language.' It appears that the notion of judgement, other than the notion of proposition, is not fully semantic in its nature, but has also a substantial pragmatic aspect. This pragmatic aspect is that what Frege expresses in his judgement stroke and what speech act theory identified as the illocutionary role or force of an assertoric act: the beholding of truth. Despite the truth predicates in \in_T -Logic and the fact that it obeys the Tarski biconditionals, the beholding of truth is not a feature of the language but an element of its use and as such a consequence of the choice of the universe of interpretation. In Martin-Löf's intuitionistic type theory this pragmatic aspect, at least in part, belongs to the 'object language,' which gives it the pragmatic flavour, expressed in the question "What can I write?"

3. One can generally say that that which is expressed in a judgement is the truth of some form of predication. This is obvious in Aristoteles' conception and in his choice of sentences, which have the valid form of a 'judgement,' and also, at least formally, in Frege's conception of a concept (in German *Begriff*) as a function whose application results a truth-value. In Brentano's conception that which is expressed in a judgement is the existence or non-existence of an object, which in Twardowski's setting is the judgements content. Existence of an object, however, can only be seen as a form of predication if the object can be represented as a matter of affairs. The situation in Martin-Löf's INTUITIONISTIC TYPE THEORY is different. That what is expressed in a judgement is the provability of a proposition, or, in a different reading, the membership in a set. There is no notion of truth but at the level of judgements in the correctness of the application of the rules. In what is expressed in a judgement, \in_T -Logic is not different to conventional formal logics, with the difference that the predications of truth and falseness differ slightly in their form.

4. If we ask, what justifies a judgement, major differences can be found. In Aristoteles' naturalistic conception justification comes objectively from the things and concerns the question of connectedness. Truth applies to thoughts as mental states and depends on a proper correspondence to the reality of things. In Brentano's conception justification has an epistemic nature and is obtained either from deductions or from inductive proofs. A different view is taken by Frege who sees the justification of a judgement to rest on necessity, which, according to him, corresponds to deduction, or empirical intuition. But truth, in his conception, is found *through* judgements, a conception, which gives the judgement stroke not only a pragmatic aspect but, other than it appeared at first, turns it at the same time into a constituent of semantics. Despite similarities in the role of judgements, Frege's view differs in this respect from the conception of Martin-Löf, who sees the basis for justification in the system of rules and not in the beholding of truth. The judgement stroke in his conception is part of pragmatics and not of semantics. In

a Tarski-style of semantics, as it is applied in \in_T -Logic and other conventional logics, the justification comes from the choice of the semantic domain in the interpretation and from the correct application of the interpretation rules. This is not much different to Frege's view, since the choice of the semantic domain of interpretation is also a judgement, and therefore not fully free from subjective influence - but other than in Frege's conception, it avoids, so to say, the responsibility for this choice to be part of the interpretation. In the view of \in_T -Logic and other conventional logics, truth and the conditions for the justification of judgements can be said to be defined. They are only found in a defined context.

Acknowledgements: This paper is devoted to Hans-Jörg Kreowski on the occasion of his 60th birthday. With Hans-Jörg I shared an office from the mid seventies to the beginning of the eighties at the Automata and Formal Languages Group at TU Berlin. What remained from these times is a feeling of friendship and trust.

I thank Andrea Hillenbrand and Sebastian Bab for discussions and for their support.

Bibliography

- [Ari67a] Aristoteles. *Analytica Priora*, I 27. 43 a 25. In *Adolf Trendelenburg, Elemente der Aristotelischen Logik - griechisch und deutsch*, p. 11 (translation by the author). Rowohlt, 1967.
- [Ari67b] Aristoteles. *De Interpretatione*, 4.17 a I. In *Adolf Trendelenburg, Elemente der Aristotelischen Logik - griechisch und deutsch*, p. 7 (translation by the author). Rowohlt, 1967.
- [Ari67c] Aristoteles. *Metaphysica*, IX 10. 1051 b 3. In *Adolf Trendelenburg, Elemente der Aristotelischen Logik - griechisch und deutsch*, p. 7 (translation by the author). Rowohlt, 1967.
- [Aus02] J. L. Austin. *Zur Theorie der Sprechakte (How to do things with Words)*. Reclam Stuttgart, 2002.
- [Bab07] S. Bab. *\in_μ -Logik – Eine Theorie propositionaler Logiken*. Shaker Verlag Aachen, 2007.
- [Ber92] J. Berg. *Ontology without Ultrafilters and Possible Worlds - An examination of Bolzano's Ontology*. Academia Sankt Augustin, 1992.
- [Bol81] B. Bolzano. *Wissenschaftslehre, Band 1 - 4*. Scientia Verlag Aalen, 1981.
- [Boo58] G. Boole. *Investigation of The Laws of Thought On Which Are Founded the Mathematical Theories of Logic and Probabilities*. Dover New York, 1958.
- [Bre95] F. Brentano. *Psychology from an Empirical Standpoint, edited by Linda L. McAlister*. Routledge London, 1995.

- [Bre08] F. Brentano. *Psychologie vom Empirischen Standpunkte - Von der Klassifikation psychischer Phänomene*. Ontos Verlag Heusenstamm, 2008.
- [Dau79] J. W. Dauben. *Georg Cantor. His Mathematics and Philosophy of the Infinite*. Princeton University Press, 1979.
- [Dum82] M. Dummett. *Wahrheit*. Reclam Stuttgart, 1982.
- [EMC⁺01] H. Ehrig, B. Mahr, F. Cornelius, M. Große-Rhode, P. Zeitz. *Mathematisch-strukturelle Grundlagen der Informatik*. Springer Berlin/Heidelberg, 2nd edition, 2001.
- [Fre66] G. Frege. Gottlob Frege - Logische Untersuchungen. In Patzig (ed.), *Logische Untersuchungen - Gedankengefüge*. Pp. 72–91. Vandenhoeck und Ruprecht Göttingen, 1966.
- [Fre75] G. Frege. Funktion und Begriff. In Patzig (ed.), *Funktion, Begriff, Bedeutung*. Vandenhoeck und Ruprecht Göttingen, 1975.
- [Fre07] G. Frege. Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens. In *Begriffsschrift und andere Aufsätze*. Georg Olms Verlag Hildesheim, 2007.
- [Gro92] R. Grossmann. *The Existence of the World - An Introduction to Ontology*. Routledge London, 1992.
- [Han98] S. O. Hansson (ed.). *Theoria - A Swedish Journal of Philosophy*. Volume 64, issues 2–3. Wiley Interscience, 1998.
- [Her83] H. Hermes. Zur Begriffsschrift und zur Begründung der Arithmetik. In *Gottlob Frege: Nachgelassene Schriften*. Meiner Verlag Hamburg, 1983.
- [HHP93] R. Harper, F. Honsel, G. Plotkin. A Framework for Defining Logics. *Journal of the ACM* 40(1), 1993.
- [Hus93] E. Husserl. *Logische Untersuchungen*. Max Niemeyer Verlag Tübingen, 1993.
- [Kan90] I. Kant. *Kritik der Urteilskraft*. Meiner Hamburg, 1990.
- [Krä88] S. Krämer. *Symbolische Maschinen - Die Idee der Formalisierung in geschichtlichem Abriss*. Wissenschaftliche Buchgemeinschaft Darmstadt, 1988.
- [Lew09] S. Lewitzka. \in_I : an intuitionistic logic without Fregean Axiom and with predicates for truth and falsity. To appear, 2009.
- [Mah93] B. Mahr. Applications of type theory. In *Proceedings of the International Joint Conference CAAP/FASE on Theory and Practice of Software Development*. Pp. 343–355. Springer Verlag, 1993.

- [Mah97] B. Mahr. Gegenstand und Kontext - Eine Theorie der Auffassung. In *K. Eyferth and B. Mahr and R. Posner and F. Wysotzki (eds.): Prinzipien der Kontextualisierung*. Pp. 101–119. KIT Report 141, Technische Universität Berlin, 1997.
- [Mah09] B. Mahr. *Die Informatik und die Logik der Modelle*, Informatik Spektrum. Volume 32(3), pp. 228–249. Springer Verlag Heidelberg, 2009.
- [Mah10] B. Mahr. Intentionality and Modelling of Conception. In Bab and Robering (eds.), *Judgements and Propositions*, Logische Philosophie Bd. 21. Logos Verlag Berlin, 2010.
- [ML84] P. Martin-Löf. *Intuitionistic Type Theory*. Bibliopolis Napoli, 1984.
- [ML96] P. Martin-Löf. *On the meaning of the Logical Constants and the Justifications of the Logical Laws*, *Nordic Journal of Philosophical Logic*, volume 1, no. 1. Scandinavian University Press, 1996.
- [ML08] P. Martin-Löf. Workshop on *Judgements, Assertions, and Propositions - The Logical Semantics and Pragmatics of Sentences* at TU Berlin. January 11, 2008.
- [Mei02] A. Meinong. *Über Annahmen*. Johann Ambrosius Barth Leipzig, 1902.
- [MSU90] B. Mahr, W. Sträter, C. Umbach. *Fundamentals of a theory of types and declarations*. Forschungsbericht, KIT-Report 82, Technische Universität Berlin, 1990.
- [vOQ80] W. van Orman Quine. *Wort und Gegenstand*. Reclam Stuttgart, 1980.
- [Par69] Parmenides. *Vom Wesen des Seienden - Die Fragmente*. Suhrkamp Frankfurt a. M., 1969.
- [Rol08] R. D. Rollinger. *Austrian Phenomenology: Brentano, Husserl, Meinong, and Others on Mind and Object*. Ontos Frankfurt, 2008.
- [Rus92] B. Russell. *Theory of Knowledge - The 1913 Manuscript*. Routledge London, 1992.
- [Rus00] B. Russell. Über das Kennzeichnen. In *Philosophische und politische Aufsätze*. Reclam Stuttgart, 2000.
- [Sea04] J. R. Searle. *Intentionality - An Essay in the Philosophy of Mind*. Cambridge University Press, 2004.
- [Sea08] J. R. Searle. *Sprechakte - Ein sprachphilosophischer Essay*. Suhrkamp Frankfurt a. M., 2008.
- [Smi89] B. Smith. Logic and the Sachverhalt. *The Monist* 72(1):52–69, Jan. 1989.
- [Smi90] B. Smith. Towards a History of Speech Act Theory. 1990.
- [Str92] W. Sträter. \in_T Eine Logik erster Stufe mit Selbstreferenz und totalem Wahrheitsprädikat. KIT-Report 98, 1992. Dissertation, Technische Universität Berlin.

- [Sun94] G. Sundholm. Existence, Proof and Truth-Making: A Perspective on the Intuitionistic Conception of Truth. *Topoi* 13:117–126, 1994.
- [Tug03] E. Tugendhat. *ti kata tinos - Eine Untersuchung zu Struktur und Ursprung Aristotelischer Grundbegriffe*. Alber Symposion Freiburg/München, 4th edition, 2003.
- [TW04] E. Tugendhat, U. Wolf. *Logisch-semantische Propädeutik*. Reclam Stuttgart, 2004.
- [Twa77] K. Twardowski. *On the Content and Object of Presentations - A Psychological Investigation*. The Hague: Martinus Nijhoff, 1977. Translated and introduced by R. Grossmann.
- [Twa82] K. Twardowski. *Zur Lehre vom Inhalt und Gegenstand der Vorstellungen - Eine psychologische Untersuchung*. Philosophia Verlag München/Wien, 1982.
- [Wie09] T. Wieczorek. *On Foundational Frames for Formal Modelling. Sets, ε -Sets and a Model of Conception*. Shaker Verlag Aachen, Dissertation, TU Berlin (Dezember 2008), 2009.
- [Wit67] L. Wittgenstein. *Philosophische Untersuchungen (Philosophical Investigations)*. Suhrkamp Frankfurt a. M., 1967.
- [Wit73] L. Wittgenstein. *Tractatus Logico Philosophicus*. Suhrkamp Frankfurt a. M., 1973.
- [WR62] A. N. Whitehead, B. Russell. *Principia Mathematica*. Cambridge University Press, 1962.
- [Zei00] P. Zeitz. *Parametrisierte \in_T -Logik: Eine Theorie der Erweiterung abstrakter Logiken um die Konzepte Wahrheit, Referenz und klassische Negation*. Logos Verlag Berlin, 2000. Dissertation, Technische Universität Berlin, 1999.